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EFFECTS OF COUPLE STRESSES IN FLUIDS ON THE DISPERSION OF A SOLUBLE MATTER IN A CHANNEL FLOW WITH HOMOGENEOUS AND HETEROGENEOUS REACTIONS

V. M. SOUNDALGEKAR*
Department of Mathematics, Indian Institute of Technology, Powai, Bombay (400076), India

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Abstract—An analysis of the dispersion of a solute in a liquid is carried out in the presence of the couple stresses, first-order, irreversible, homogeneous and the heterogeneous chemical reaction. In case of homogeneous chemical reaction, couple stresses are more effective at small values of γ, the homogeneous reaction rate parameter. But at large γ, the Taylor diffusion coefficient D^* decreases with increasing α, the couple stress parameter. Also, owing to the presence of the heterogeneous chemical reaction, there is a sharp rise in D^*.

NOMENCLATURE

C,	concentration of the solute;
D,	molecular diffusion coefficient;
D^*,	Taylor diffusion coefficient;
h,	half-width of the channel;
K_1,	first-order reaction rate constant;
l,	defined as $(\eta_1/\mu)^{1/2}$;
L,	characteristic length;
P,	pressure gradient;
Q,	volume rate of the transport of the solute;
$\bar{t}$,	time $(L/\bar{u})$;
t',	time;
u_x,	axial velocity;
$\bar{u}$,	average velocity;
v_x,	relative velocity;
η_1,	material constant with the dimension (ML/T).

Greek symbols

α,	defined as $(= h/l)$;
ξ,	non-dimensional axial co-ordinate axis $(x - \bar{u}t/L)$;
η,	non-dimensional transverse co-ordinate axis (y/h);
γ,	non-dimensional number $[h(K_1/D)^{1/2}]$;
β,	heterogeneous reaction rate parameter (fh);
θ,	non-dimensional time.

1. INTRODUCTION

TAYLOR [1–3] in a series of papers discussed the dispersion of a soluble matter in a viscous incompressible fluid flow in a circular pipe. This was extended by Aris [4] who showed that the rate of the growth of the variance of the solute distribution is proportional to the sum of the molecular diffusion coefficient and the Taylor diffusion coefficient. From the technological point of view, these problems are always important. But in the above papers, it was assumed that there is no chemical reaction of any kind between the solute and the fluid during the course of dispersion. This was not always true. In many physical phenomenon, homogeneous and heterogeneous chemical reactions are always present. Such an analysis of a finite first-order homogeneous reaction in a laminar pipe flow was first studied both theoretically and experimentally by Cleland and Wilhelm [5]. The effect of the heterogeneous reaction taking place at the wall was discussed by Katz [6], whereas Walker [7], Soloman and Hudson [8] investigated the combined effect of the first-order heterogeneous and homogeneous reactions. Recently Gupta and Gupta [9] extended this study to investigate the effects of both homogeneous and heterogeneous chemical reactions on the dispersion of soluble matter in a parallel plate channel flow.

In all these papers, the fluid considered was a Newtonian fluid whose motion is governed by the Navier–Stokes equations which does not assume the presence of the couple stresses in fluids. The governing equations on taking into account the couple stresses were derived by Stokes [10] and studied the effects of the couple stresses on the channel flow. The effects of the couple stresses on the dispersion of the solute in a channel flow, in the absence of the chemical reactions, was presented by Soundalgekar [11]. It is now the object of this paper to study the effects of the couple stresses on the dispersion of the soluble matter in an

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*Present address: Mechanical Engineering UWIST, Cardiff, U.K.

incompressible viscous fluid in the presence of the homogeneous and heterogeneous chemical reactions.

In Section 2, the expression for the velocity profile in a channel flow, as derived by Stokes is assumed and the expressions for the effective Taylor diffusion coefficient are derived in case of (i) the first-order homogeneous, irreversible chemical reaction and (ii) heterogeneous reaction at the catelytic walls. This is followed by a discussion.

2. MATHEMATICAL ANALYSIS

(i) *Homogeneous first-order reaction*

The expression for the velocity profile for a laminar flow of a viscous fluid between two parallel plates $y = \pm h$, as derived by Stokes [10] is

$$u_x(y) = -\frac{h^2P'}{2\mu}\left[1-\eta^2-\frac{2}{\alpha^2}\left(1-\frac{\cosh\alpha\eta}{\cosh\alpha}\right)\right] \tag{1}$$

where P' is the pressure gradient along the plates, μ is the viscosity, $\alpha = h/l$, $l = (\eta_1/\mu)^{1/2}$ and η_1 is a material constant with the dimensions (ML/T). This shows that the parameter α depends upon h, half-width of the channel.

The average velocity is given by

$$\bar{u} = \frac{1}{2h}\int_{-h}^{h} u_x\,\mathrm{d}y. \tag{2}$$

From (1) and (2), we have

$$\bar{u} = Z\left[\frac{2}{3}-\frac{2}{\alpha^2}\left(1-\frac{\tanh\alpha}{\alpha}\right)\right] \tag{3}$$

where $Z = -h^2P'/2\mu$.

The governing equation for the concentration C of the solute diffusing in the fluid in the presence of a first-order homogeneous irreversible chemical reaction is

$$\frac{\partial C}{\partial t} + u_x\frac{\partial C}{\partial x} = D\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right) - K_1 C \tag{4}$$

where D is the molecular diffusion, assumed constant and K_1 is the first-order reaction rate constant. It is assumed in (4) that the solute is present in a small concentration and the last term $-K_1C/\mathrm{mol\,cm^{-3}s^{-1}}$ represents the volume rate of the disappearance of the solute due to chemical reaction. We also assume that $\partial^2C/\partial x^2 \ll \partial^2C/\partial y^2$ and that the convection is across a plane moving with the mean speed of the flow. Relative to this plane the velocity of the fluid is derived from (1) and (3) as

$$v_x = u_x - \bar{u} = Z\left[\frac{1}{3}-\eta^2-\frac{2}{\alpha^2}\left(\frac{\tanh\alpha}{\alpha}-\frac{\cosh\alpha\eta}{\cosh\alpha}\right)\right] \tag{5}$$

With these assumptions, equation (4) reduces to

$$\frac{1}{\bar{t}}\frac{\partial C}{\partial \theta} + \frac{v_x}{L}\frac{\partial C}{\partial \xi} = \frac{D}{h^2}\frac{\partial^2 C}{\partial \eta^2} - K_1 C \tag{6}$$

where L is a given length along the flow direction and

$$\theta = t/\bar{t},\quad \bar{t} = L/\bar{u},\quad \xi = \frac{x-\bar{u}t}{L},\quad \eta = y/h. \tag{7}$$

Assuming the validity of Taylor's limiting condition, viz. the partial equilibrium along any cross-section of the channel, then (6) reduces to

$$\frac{\partial^2 C}{\partial \eta^2} - \gamma^2 C = \frac{h^2}{DL}v_x\frac{\partial C}{\partial \xi} \tag{8}$$

where $\gamma = h(K_1/D)^{1/2}$.

Inserting for v_x from (5) in (8), and solving under the boundary conditions $\partial C/\partial\eta = 0$ at $\eta = \pm 1$, we have

$$C = A\cosh\gamma\eta + \frac{h^2 Z}{DL}\frac{\partial C}{\partial \xi} \times\left\{-\frac{1}{3\gamma^2}+\frac{\eta^2}{\gamma^2}+\frac{2}{\gamma^4}+\frac{2}{\alpha^2} \times\left[\frac{\tanh\alpha}{\gamma^2\alpha}+\frac{\cosh\alpha\eta}{(\alpha^2-\gamma^2)\cosh\alpha}\right]\right\} \tag{9}$$

where

$$A = -\frac{h^2Z}{DL}\frac{\partial C}{\partial \xi}\frac{2}{\gamma\sinh\gamma}\left[\frac{1}{\gamma^2}+\frac{\tanh\alpha}{\alpha(\alpha^2-\gamma^2)}\right].$$

Now the volume rate of the transport of the solute across a section of the channel of unit breadth is given by

$$Q = h\int_{-1}^{1} Cv_x\,\mathrm{d}y. \tag{10}$$

Using (5) and (9), we obtain from (10),

$$\begin{aligned} Q = \frac{h^7P'^2}{\mu DL}\frac{\partial C}{\partial \xi}&\left\{\frac{2}{\gamma\sinh\gamma}\left[\frac{1}{\gamma^2}+\frac{\tanh\alpha}{\alpha(\alpha^2-\gamma^2)}\right]\right\} \\ &\times\left\{\frac{\cosh\gamma}{\gamma^2}-\frac{\sinh\gamma}{\gamma^3}-\frac{\tanh\alpha\sinh\gamma}{2\alpha^3\gamma}-\frac{\sinh\gamma}{3\gamma}\right. \\ &\left.+\frac{1}{2\alpha^2\cosh\gamma}\left[\frac{\sinh(\alpha+\gamma)}{\alpha+\gamma}+\frac{\sinh(\alpha-\gamma)}{\alpha-\gamma}\right]\right\} \\ &-\frac{2}{45\gamma^2}+\frac{\tanh\alpha}{3\alpha^3}\cdot\frac{(\alpha^2-2\gamma^2)}{\gamma^2(\alpha^2-\gamma^2)}-\frac{(\alpha^2+2\gamma^2)\tanh\alpha}{3\alpha^3\gamma^2(\alpha^2-\gamma^2)} \\ &+\frac{2(1-\alpha)\tanh\alpha}{\alpha^3(\alpha^2-\gamma^2)}-\frac{2}{\alpha^3\gamma^2}\left[\frac{1}{\alpha}-\left(\frac{1}{3}+\frac{1}{\alpha^2}\right)\tanh\alpha\right] \\ &-\frac{1}{\alpha^4}\left[\frac{2\tanh^2\alpha}{\alpha^2(\alpha^2-\gamma^2)}-\frac{\sinh 2\alpha+2\alpha}{4\alpha(\alpha^2-\gamma^2)\cosh^2\alpha}\right]\ldots \end{aligned} \tag{11}$$

On comparing (11) with Fick's law of diffusion, we find that the solute is dispersed relative to a plane moving with the mean speed of the flow with an effective Taylor diffusion coefficient D^* given by

$$D^* = \frac{h^6\left(\frac{\partial P}{\partial \xi}\right)^2}{D\mu^2} F(\alpha,\gamma) \tag{12}$$

where

$$\begin{aligned} F(\alpha,\gamma) = {} & \frac{2}{\gamma\sinh\alpha}\left[\frac{1}{\gamma^2}+\frac{\tanh\alpha}{\alpha(\alpha^2-\gamma^2)}\right]\left\{\frac{\sinh\gamma}{\gamma^3}\right. \\ & +\left[\frac{\tanh\alpha\sinh\gamma}{2\alpha^3\gamma}+\frac{\sinh\gamma}{3\gamma}-\frac{\cosh\gamma}{\gamma^2}-\frac{1}{2\alpha^2\cosh\alpha}\right] \\ & \left.\times\left[\frac{\sinh(\alpha+\gamma)}{\alpha+\gamma}+\frac{\sinh(\alpha-\gamma)}{\alpha-\gamma}\right]\right\}+\frac{2}{45\gamma^2} \\ & -\frac{(\alpha^2-2\gamma^2)\tanh\alpha}{3\alpha^3\gamma^2(\alpha^2-\gamma^2)}+\frac{(\alpha^2+2\gamma^2)\tanh\alpha}{2\gamma^2\alpha^3(\alpha^2-\gamma^2)} \\ & -\frac{2(1-\alpha)\tanh\alpha}{\alpha^3(\alpha^2-\gamma^2)}+\frac{2}{\alpha^3\gamma^2}\left[\frac{1}{\alpha}-\left(\frac{1}{3}+\frac{1}{\alpha^2}\right)\tanh\alpha\right] \\ & +\frac{1}{\alpha^4}\left[\frac{2\tanh^2\alpha}{\alpha^2(\alpha^2-\gamma^2)}-\frac{\sinh 2\alpha+2\alpha}{4\alpha(\alpha^2-\gamma^2)\cosh^2\alpha}\right]. \end{aligned} \tag{13}$$

The numerical values of $F(\alpha,\gamma)$ are entered in Table 1.

Table 1. Values of $F(\alpha,\gamma)\times 10^{+3}$

	α	
γ	0·2	0·3
0·1	24	239
0·2	0·5696	2·9165

Thus, D^* is governed by two dimensionless parameters, viz. γ and α. γ can be looked upon as a reaction rate parameter. Also $\alpha = h/l$ and as l is constant for a given liquid, D^* is governed by h, the half-width of the channel. We observe from this table that at small values of γ, there is a sudden rise in D^* with increasing α. But as γ is increased, the rate of decrease in D^* is also large.

(ii) *Diffusion with combined homogeneous and heterogeneous chemical reaction*

If there is also first-order irreversible chemical reaction at the walls, which are assumed to be catalytic, when there is present the chemical reaction in the bulk of the fluid, then equation (8) still governs the diffusion-phenomenon. The boundary conditions are

$$\frac{\partial C}{\partial y} \pm fC = 0 \quad \text{at} \quad y = \pm h. \tag{14}$$

In non-dimensional form, these reduce to

$$\frac{\partial C}{\partial \eta} \pm \beta C = 0 \quad \text{at} \quad \eta = \pm 1 \tag{15}$$

where $\beta = fh$ is the heterogeneous reaction rate parameter corresponding to catalytic reaction at the walls. Proceeding as above, we can find the expression for the Taylor diffusion coefficient as

$$D^* = \frac{h^6\left(\frac{\partial p}{\partial \xi}\right)^2}{\mu D} F_1(\alpha,\beta,\gamma) \tag{16}$$

where

$$\begin{aligned} F_1(\alpha,\beta,\gamma) = {} & \frac{X}{\gamma}\left[\frac{\sinh\gamma}{3\gamma}-\frac{\cosh\gamma}{\gamma^2}+\frac{\sinh\gamma}{\gamma^3}\right. \\ & +\frac{\sinh\gamma\tanh\alpha}{\alpha^3\gamma}-\frac{\sinh(\alpha+\gamma)}{4\alpha^2(\alpha+\gamma)\cosh\alpha} \\ & \left.-\frac{\sinh(\alpha-\gamma)}{4\alpha^2(\alpha-\gamma)\cosh\alpha}\right]+\frac{2}{45\gamma^2}-\frac{\tanh\alpha}{\alpha^3\gamma^2} \\ & +\frac{2}{\alpha^4\gamma^2}-\frac{\tanh\alpha}{\alpha^5\gamma^2}+\frac{\tanh\alpha}{3\alpha^3\gamma^2}-\frac{1}{\alpha^2(\alpha^2-\gamma^2)} \\ & +\frac{(\alpha^2+1)\tanh\alpha}{\alpha^3(\alpha^2-\gamma^2)}+\frac{\sinh\alpha\tanh\alpha}{\alpha^4}-\frac{\sinh\alpha}{2\alpha^3} \\ & -\frac{1}{2\alpha^3\cosh\alpha} \end{aligned} \tag{17}$$

where

$$\begin{aligned} X = {} & \frac{2}{\gamma^2}+\frac{2\tanh\gamma}{\alpha(\alpha^2-\gamma^2)}+\beta\left[\frac{2}{3\gamma^2}+\frac{2}{\gamma^4}+\frac{2}{\alpha^2}\right. \\ & \left.\times\left(\frac{\tanh\alpha}{\gamma^2\alpha}+\frac{1}{\alpha^2-\gamma^2}\right)\right] \end{aligned}$$

$$Y = \gamma\sinh\gamma+\beta\cosh\gamma.$$

The numerical values of $F_1(\alpha,\beta,\gamma)$ are entered in Table 2.

Table 2. Values of $F_1(\alpha,\beta,\gamma)\times 10^6$

		β		
γ	α	1	2	3
0·1	0·1	2·50474	2·50483	2·50485
0·1	0·2	$6{\cdot}41948\times 10^{-2}$	$6{\cdot}41898\times 10^{-2}$	$6{\cdot}41887\times 10^{-2}$
0·2	0·1	$4{\cdot}21916\times 10^{-1}$	$4{\cdot}22979\times 10^{-1}$	$4{\cdot}29230\times 10^{-1}$
0·2	0·2	$3{\cdot}93124\times 10^{-2}$	$3{\cdot}93347\times 10^{-2}$	$3{\cdot}93394\times 10^{-2}$

We observe from this table that due to the presence of the heterogeneous chemical reaction at the walls, there is a sharp rise in D^* as compared to $\beta = 0$. An increase in α or γ leads to a decrease in D^*.

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